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Catenary deformations of inextensible networks

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Abstract. A network formed from two initially orthogonal families of cables is supported at its edges and allowed to sag under its own weight. The net is idealized as a continuum formed from inextensible fibers, with no shear resistance. Several families of exact solutions are exhibited, all involving catenaries in one way or another. These solutions may be useful in connection with perturbation analyses of boundary-value problems.

1. Introduction

This paper gives some examples of deformations of a cable network that hangs under its own weight. The network is formed from two families of cables that initially lie parallel to the x- and y-directions of a system of Cartesian coordinates. In one set of solutions, the cables of one family are congruent catenaries that lie in planes that need not be vertical, while the cables of the second family lie along arbitrary congruent curves in vertical planes. In another set, the catenaries lie in vertical planes but the cross-cables lie in planes that need not be vertical. In the final set, the deformed network is a cylinder with catenary cross-section. Other special cases are considered.

We use Rivlin's [1] theory of inextensible networks with no shear resistance (see also Pipkin [2, 3], Kuznetsov [4]). The deformations that we consider furnish exact finite-deformation solutions of the governing equations for this theory (Sec. 2). Boundary conditions are not pre-assigned, so it would be only accidental if one of these solutions should satisfy some specified set of conditions. Problems with specified boundary data are often solved by linearization about some exact non-linear solution; the families of exact solutions given here may be useful for this purpose.

2. Basic equations

We consider nets formed from inextensible fibers that initially lie parallel to the x- and y-axes of a system of Cartesian coordinates. The fibers are fixed together at the points where they intersect. The net is treated as a continuum, so that every material line $x = \text{constant}$ or $y = \text{constant}$ is regarded as an inextensible fiber. In a deformation, the particle initially at (x, y) goes to the place $\mathbf{r}(x, y)$ in three-dimensional space. The derivatives

$$\mathbf{a} = \mathbf{r}_x \quad \text{and} \quad \mathbf{b} = \mathbf{r}_y \,, \tag{2.1}$$

are vectors tangential to the deformed fibers, and we refer to these fibers as **a**-lines and **b**-lines. The assumed inextensibility of the fibers means that **a** and **b** are unit vectors:

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1 \,. \tag{2.2}$$

In all cases to be considered here, the deformed network forms a *translation surface* (Voss [5]),

$$\mathbf{r}(x, y) = \mathbf{P}(x) + \mathbf{Q}(y) , \qquad \mathbf{a} = \mathbf{P}'(x) , \qquad \mathbf{b} = \mathbf{Q}'(y) . \tag{2.3}$$

It is convenient to take $\mathbf{P}(0) = \mathbf{Q}(0) = \mathbf{0}$, so that $\mathbf{r} = \mathbf{P}(x)$ is the curve followed by the deformed fiber $y = 0$ and $\mathbf{r} = \mathbf{Q}(y)$ represents the deformed fiber $x = 0$. We call these *base curves* [3]. All **a**-lines are congruent to $\mathbf{r} = \mathbf{P}(x)$, and all **b**-lines are congruent to $\mathbf{r} = \mathbf{Q}(y)$. We usually exhibit only the forms of $\mathbf{a}(x)$ and $\mathbf{b}(y)$; $\mathbf{P}(x)$ and $\mathbf{Q}(y)$ can then be obtained by integration.

In Rivlin's [1] theory it is assumed that there is no resistance to changes in the angle between fibers of the two families. The force per unit length exerted on a **b**-line by the **a**-lines is $T_a\mathbf{a}$, and that exerted on an **a**-line by the **b**-lines is $T_b\mathbf{b}$. We refer to T_a and T_b as fiber tensions, although in fact the tension in a real **a**-line is T_a/n, where n is the number of real cables y = constant per unit length in the y-direction. The force acting from right to left across the arc $(\mathrm{d}x, \mathrm{d}y)$ then

$$T_a\mathbf{a}\,\mathrm{d}y - T_b\mathbf{b}\,\mathrm{d}x . \tag{2.4}$$

Let w be the weight of the network, per unit initial area, and let **k** be a vertically upward unit vector. Then equilibrium requires that

$$(T_a\mathbf{a})_x + (T_b\mathbf{b})_y = w\mathbf{k} . \tag{2.5}$$

The solutions that we obtain are for cases in which w is constant. The tensions T_a and T_b are reactions to the inextensibility constraints (2.2), taking whatever values equilibrium may require.

3. Independent catenaries

The simplest solutions are those in which the fibers of one family are unstressed, say

$$T_b = 0 , \quad \mathbf{b}(y) \text{ arbitrary} . \tag{3.1}$$

The shape of the **b**-lines is arbitrary since they carry no load. In such cases all of the weight of both families is supported by the **a**-lines alone. With $T_b = 0$, (2.5) yields

$$\mathbf{a}(x) = (\mathbf{a}_0 + \mathbf{k}x/L)/A(x) \tag{3.2}$$

and

$$T_a = wLA(x) , \tag{3.3}$$

where $A(x)$ is the magnitude of the numerator in (3.2), so that **a** is a unit vector. If $\mathbf{a}_0 = \pm\mathbf{k}$, the **a**-lines hang straight down. If $\mathbf{a}_0$ is not vertical, the numerator in (3.2) is horizontal for some x, so by choice of the origin of x we can take $\mathbf{a}_0$ to be a horizontal unit vector. With this choice,

$$A(x) = [1 + (x/L)^2]^{1/2} . \tag{3.4}$$

L is an arbitrary length.

By integrating $\mathbf{P}'(x) = \mathbf{a}(x)$ we find that

$$\mathbf{P}(x)/L = \mathbf{a}_0\zeta + \mathbf{k}\cosh\zeta\ , \tag{3.5}$$

where

$$\sinh\zeta = x/L\ . \tag{3.6}$$

The coordinate x measures arc length along the deformed or undeformed fiber. In terms of horizontal and vertical coordinates $\xi = \mathbf{a}_0 \cdot \mathbf{P}$ and $\eta = \mathbf{k} \cdot \mathbf{P}$, (3.5) is equivalent to $\eta/L = \cosh(\xi/L)$, so the curve $\mathbf{r} = \mathbf{P}(x)$ is a catenary. Just as a single cable in isolation hangs as a catenary, the **a**-lines here are all catenaries because the **b**-lines exert no force on them except the part of the weight that is due to the **b**-lines.

4. Cables in vertical planes

Now let the **a**-lines be catenaries, with $\mathbf{a}(x)$ given by (3.2) and (3.4), and let the base curve $\mathbf{r} = \mathbf{Q}(y)$ for the second family be an arbitrary curve in a vertical plane. Then $\mathbf{b}(y)$ has the form

$$\mathbf{b}(y) = [\mathbf{b}_0 + f(y)\mathbf{k}]/B(y)\ , \tag{4.1}$$

where

$$B(y) = [1 + f^2(y)]^{1/2}\ . \tag{4.2}$$

Here $\mathbf{b}_0$ is an arbitrary horizontal unit vector and $f(0) = 0$ by convention. (The case in which the **b**-lines hang straight down can be treated separately.)

It is easy to verify that (2.5) is satisfied if

$$T_a = [w - Cf'(y)]LA(x) \tag{4.3}$$

and

$$T_b = CB(y)\ , \tag{4.4}$$

where C is arbitrary. The deformation is stable only if the fiber tensions are non-negative, so $C \geqslant 0$ is required for stability. Assuming that $C > 0$ and $L > 0$ (so that the **a**-lines cup upward), we see that the tension in the **b**-lines relieves some of the tension in the **a**-lines if $f'(y) > 0$ and adds to the tension in the **a**-lines if $f'(y) < 0$. In the former case the **b**-lines cup upward, and they cup downward in the latter case.

When $f(y) = y/L_1$, the **b**-lines are catenaries too. Let p be defined by $C = wL_1p$. Then with both families vertical catenaries, the tensions are

$$T_a = (1 - p)wLA(x)\ , \qquad T_b = pwL_1B(y)\ , \tag{4.5}$$

where

$$B(y) = [1 + (y/L_1)^2]^{1/2}\ . \tag{4.6}$$

If L and L_1 are positive, the tensions are non-negative if $0 \leqslant p \leqslant 1$. We see that p represents the proportion of the total weight that is supported by the **b**-lines.

When $w = 0$ but $C \neq 0$ the solution is a special case of a general class of solutions found in an earlier paper [3] in which all equilibrium configurations were determined for weightless networks deformed as translation surfaces.

5. Swayed cross-cables

Let the **a**-lines be catenaries in vertical planes, as in Section 4, but suppose that the base curve $\mathbf{r} = \mathbf{Q}(y)$ is an arbitrary plane curve in a plane that is not vertical. Let $\mathbf{b}(0) = \mathbf{b}_0$, and suppose that $\mathbf{r} = \mathbf{Q}(y)$ lies in the plane spanned by $\mathbf{b}_0$ and $\mathbf{k} + \alpha\mathbf{a}_0$. Then $\mathbf{b}(y)$ has the form

$$\mathbf{b}(y) = [\mathbf{b}_0 + f(y)(\mathbf{k} + \alpha\mathbf{a}_0)]/B(y)\,, \tag{5.1}$$

where $f(0) = 0$. $B(y)$ is the magnitude of the vector in the numerator. Recall that $\mathbf{a}(x)$ is given by (3.2) and (3.4). Then it is straightforward to verify that (2.5) is satisfied if

$$T_a = LA(x)\left[w - \frac{Cf'(y)}{1 - \alpha x/L}\right] \tag{5.2}$$

and

$$T_b = CB(y)/(1 - \alpha x/L)^2\,. \tag{5.3}$$

Because of the singularities at $x = L/\alpha$, the domain of x must be restricted to one of the regions $x > L/\alpha$ or $x < L/\alpha$. At $x = L\alpha$, $\mathbf{a}(x)$ is parallel to $\mathbf{k} + \alpha\mathbf{a}_0$ and thus the **a**-line is tangential to the plane of the **b**-line.

When $f(y) = y/L_1$, the **b**-lines are also catenaries. We show how to verify this in a similar case in Section 6. For this case, write C as $C = wL_1p$. Then

$$T_a = w[(1-p)L - \alpha x]A(x)/(1 - \alpha x/L) \tag{5.4}$$

and

$$T_b = pwL_1B(y)/(1 - \alpha x/L)^2\,. \tag{5.5}$$

If L and L_1 are positive, $p \geqslant 0$ is needed to ensure that $T_b \geqslant 0$. Then $T_a \geqslant 0$ as well in the two separate regions $\alpha x < (1-p)L$ and $\alpha x > L$.

6. Swayed catenaries

Now let the **b**-lines be congruent to an arbitrary curve $\mathbf{r} = \mathbf{Q}(y)$ in a vertical plane, as in Section 4, but suppose that the catenary **a**-lines are swayed out of the vertical. Then $\mathbf{a}(x)$ has the form

$$\mathbf{a}(x) = [\mathbf{a}_0 + (x/L)(\mathbf{k} + \beta\mathbf{b}_0)]/A(x)\,, \tag{6.1}$$

where $\mathbf{a}_0$ and $\mathbf{b}_0$ are horizontal unit vectors and $A(x)$ is the magnitude of the numerator, so that $\mathbf{a}(x)$ is a unit vector. To see that the $\mathbf{a}$-lines are indeed catenaries, define

$$\mathbf{v} = \mathbf{k} + \beta \mathbf{b}_0 , \qquad v^2 = 1 + \beta^2 , \tag{6.2}$$

$$\mathbf{u} = \mathbf{a}_0 - \mathbf{v}\beta \mathbf{a}_0 \cdot \mathbf{b}_0 / v^2 , \tag{6.3}$$

and

$$\bar{x}/L = x/L + \beta \mathbf{a}_0 \cdot \mathbf{b}_0 / v^2 . \tag{6.4}$$

The magnitude $u = |\mathbf{u}|$ is given by

$$u^2 = 1 - (\beta \mathbf{a}_0 \cdot \mathbf{b}_0 / v)^2 . \tag{6.5}$$

Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, and (6.1) takes the form

$$\mathbf{a}(x) = (\mathbf{u} + \mathbf{v}\bar{x}/L)/A(x) \tag{6.6}$$

with

$$A(x) = [u^2 + (v\bar{x}/L)^2]^{1/2} . \tag{6.7}$$

Integration of $\mathbf{P}'(x) = \mathbf{a}(x)$ then gives

$$\mathbf{P}(x) = (Lu/v)[(\mathbf{u}/u)\zeta + (\mathbf{v}/v)\cosh \zeta] , \tag{6.8}$$

where

$$\sinh \zeta = (v/u)(\bar{x}/L) . \tag{6.9}$$

With $\mathbf{a}(x)$ given by (6.1) and $\mathbf{b}(y)$ by (4.1), it can be verified that (2.5) is satisfied if

$$T_a = [w(1 - \beta f + \beta y f') - Cf'(y)]LA(x)/(1 - \beta f)^2 \tag{6.10}$$

and

$$T_b = (C - \beta w y)B(y)/(1 - \beta f) , \tag{6.11}$$

where $A(x)$ is given by (6.7) and $B(y)$ by (4.2). The constant C is arbitrary.

In the special case $f(y) = y/L_1$, the $\mathbf{b}$-lines are catenaries and the expressions (6.10) and (6.11) reduce to forms like (5.4) and (5.5), but with the roles of $\mathbf{a}$- and $\mathbf{b}$-lines interchanged.

7. Catenary cylinders

When $\mathbf{b}(y) = \mathbf{b}_0$, constant, the $\mathbf{b}$-lines are parallel straight lines and the deformed surface is a cylinder. It is found that for equilibrium, $\mathbf{a}(x)$ must have the form

$$\mathbf{a}(x) = [\mathbf{a}_0 + (x/L)(\mathbf{k} + \beta(x)\mathbf{b}_0)]/A(x) , \tag{7.1}$$

where $A(x)$ is the magnitude of the vector in the numerator. Except for the term involving

$\mathbf{b}_0$, $\mathbf{a}(x)$ is parallel to $\mathbf{a}_0 + \mathbf{k}x/L$, so the cross-section of the cylinder is a catenary. The $\mathbf{a}$-lines are catenaries themselves if $\beta(x)$ is constant, but otherwise they are rather arbitrary (congruent) curves on the cylinder.

The tensions are found from (2.5) to be

$$T_a = wLA(x) \tag{7.2}$$

and

$$T_b = T_0(x) - wy[x\beta(x)]' . \tag{7.3}$$

Here $T_0(x)$ represents an arbitrary tension in each straight $\mathbf{b}$-line.

It is possibly worth pointing out that in all of the preceding solutions, the unit vector $\mathbf{b}_0$ can be taken to lie in the plane of $\mathbf{a}_0$ and $\mathbf{k}$. The deformed net is then all in one vertical plane. In such cases it can happen that some part of the plane is covered more than once by the net. For the real physical network, this means that two parts of the network are side by side in contact.

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